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Periodic structures with Rashba interaction in a magnetic field

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Abstract

We analyze the behavior of a system of particles living on a periodic crystal in the presence of a magnetic field B. This can be done by involving a periodic potential $U(x)$ and the Rashba interaction of the coupling constant k_{so}. By resorting to the corresponding spectrum, we explicitly determine the band structures and the Bloch spinors. These allow us to discuss the system symmetries in terms of the polarizations where they are shown to be broken. The dynamical spin will be studied by calculating different quantities. In the limits: k_{so} and $U(x) = 0$, we analyze again the system by deriving different results. Considering the strong B case, we obtain an interesting result that is the conservation of the polarizations. Analyzing the critical point $\lambda_{k,\sigma} = \pm\sqrt{\frac{1}{2}}$, we show that the Hilbert space associated with the spectrum in the z-direction has a zero mode energy similar to that of massless Dirac fermions in graphene. Finally, we give the resulting energy spectrum when $B = 0$ and $U(x)$ is arbitrary.

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1. Introduction

Unlike the quantum Hall effect (QHE) [1] that can be achieved from the impurities [2] or the Coulomb interaction [3], the spin Hall effect (SHE) [4] is essentially due to something else. In fact, this phenomena can be seen as a consequence of the Rashba or Dresselhaus interactions [5–7]. They depend on what kind of spin–orbit coupling is involved and more precisely they are originated from two different effects. The first one is caused by the asymmetry of the confining quantum well but the second is a bulk effect [8]. The spin–orbit coupling leads to a spin accumulation in the direction transverse to the applied electric field. These show the importance of such interactions and motivate their analysis in different contexts.

Many theories appeared as good candidates to describe different issues related to the Rashba interaction. Different questions have been addressed and lead to interesting developments on the subject. In particular analytical and numerical results for the effect of such spin–orbit coupling on band structure, transport and interaction effects in quantum wires, when the spin precession length is comparable to the wire width, have been reported in [9]. The Harper–Hofstadter problem for two-dimensional electron gas with Rashba interaction subject to periodic potential and perpendicular magnetic field has been studied analytically and numerically [10]. One may also cite other interesting papers related to the subject such as [11–14].

Very recently, an analysis has appeared dealing with other issues. This concerns the investigations of the basic features of a two-dimensional electron gas with the Rashba interaction and two in-plane potentials superimposed along directions perpendicular to each other [15]. The first of these potentials is assumed to be a general periodic potential while the second one is totally arbitrary. A general form for Bloch's amplitude is found and an eigenvalue problem for the band structure of the system is derived. Moreover the general result is applied to the two particular cases in which either the second potential represents a harmonic in-plane confinement or it is zero. It is found that for a harmonic confinement regions of the Brillouin zone with high polarizations are associated with those of large group velocity.

Motivated by the investigations of different groups cited above and in particular [15], we develop our proposal to deal with different issues. For this, we consider a system of particles living on the plane (x, z) in the presence of an external magnetic field. To do our task, we include a period potential $U(x)$ as well as the Rashba interaction. By splitting the corresponding Hamiltonian, we diagonalize different parts to end up with the total eigenvalues and Bloch spinors for the two first transverse modes, i.e $n = 0, 1$. These will be used to show that there is a hidden symmetry in the spectrum.

On one hand, by analyzing the polarizations we conclude that such symmetries are not preserved for the present system. Moreover, we study the dynamical spin by evaluating the velocity and spin current operator, which are used to give the corresponding Hall conductivity. Moreover, we discuss two limits where the first is the Rashba coupling constant and this leads to the Landau problem being submitted to the periodicity constraint in the x-direction. The second one is $U(x) = 0$ that allows us to get the same Landau problem but amended with the Rashba interaction.

To deeply investigate the basic features of the present system, we treat an interesting case. Indeed, for a strong B we analyze the corresponding spectrum and symmetries to end up with different conclusions. As an interesting consequence, we show that the polarization symmetries are preserved in such cases and this generalize the obtained result in [15] to the magnetic field. More importantly, such polarizations are obtained to be B-dependent and therefore it can be controlled by different values taken by the magnetic field.

On the other hand, we consider the critical point $\lambda_{k,\sigma} = \pm\sqrt{\frac{1}{2}}$ that appears as a singularity of the spectrum to underline its influence on the obtained results. More precisely, by resorting to different quantities we note the spectrum corresponding to the z-direction is sharing some common features with the graphene. In particular, a zero mode energy will come up in a similar way to massless Dirac fermions. In the end, we analyze the periodic structures when B is switched off and by considering two limits we recover some known results such as the Rashba dispersion relation.

The present paper is organized as follows. In section 2, for later convenience, we study the Bloch theory for the one-dimensional system and establish the Hamiltonian for the two-dimensional system where the Rashba interaction is included. This will be generalized by introducing a magnetic field B perpendicular to the periodic crystal in section 3 in order to create a confinement in one direction. Moreover, we split the corresponding Hamiltonian into two parts to get an easy diagonalization. In section 4, we treat different parts to obtain the eigenvalues and eigenstates and for the total spectrum we restrict ourselves to the two first modes. In section 5, we consider two different limits: $k_{so} = 0$ and $U(x) = 0$ to characterize the system behavior in such cases. We analyze the polarization symmetries by evaluating different components in section 6 and show that only those in the z-direction are broken whereas the remaining are nulls. We discuss the spin dynamical properties by evaluating some relevant quantities in section 7. We consider the case when B is strong to deduce different conclusions in section 8. The critical point will be analyzed in section 9. We also consider the periodic structure of the system when B is switched off in section 9. Finally, we conclude our work.

2. Bloch theory for the one-dimensional system

We start by showing the spinorial structure influence on the one-dimensional periodic crystal. This can be done by making use of a Bloch theory and characterizing the system behavior. In doing so, let us consider a simple Hamiltonian, which is given by

$$H_0^{\mathrm{1D}} = \frac{\hbar^2 k_x^2}{2m} + U(x) \tag{1}$$

where $U(x)$ is a periodic potential, such as

$$U(x + L) = U(x) \tag{2}$$

and L is its period along the x-direction. Clearly, without $U(x)$ we simply have a plane wave in one dimension as a solution of the system.

Note that H_0^{1D} is a simple problem that can easily be solved. Indeed, the eigenvalues $\varepsilon_{l,\sigma}^{(0)}(k_B)$ and eigenstates $|l, k_B, \sigma\rangle$ are solutions of the equation

$$H_0^{\mathrm{1D}}|l, k_B, \sigma\rangle = \varepsilon_{l,\sigma}^{(0)}(k_B)|l, k_B, \sigma\rangle. \tag{3}$$

They are labeled by the Bloch's quasi-momentum k_B that is a discrete value in the Brillouin zone and the band index l. The energies $\varepsilon_{l,\sigma}^{(0)}(k_B)$ are degenerates with respect to the spin index because one can obtain the relations

$$\varepsilon_{l,+1}^{(0)}(k_B) = \varepsilon_{l,-1}^{(0)}(k_B) \equiv \varepsilon_l^{(0)}(k_B) \tag{4}$$

where $\varepsilon_l^{(0)}(k_B)$ reads as

$$\varepsilon_l^{(0)}(k_B) = \frac{\hbar^2 k_B^2}{2m}. \tag{5}$$

To get the corresponding Bloch wavefunctions, we simply project the states $|l, k_B, \sigma\rangle$ on the coordinate representation $\{|x, \sigma\rangle\}$. Performing this process to get

$$\langle x, \sigma'|l, k_B, \sigma\rangle = \frac{1}{\sqrt{L_0}} \mathrm{e}^{\mathrm{i}k_B x} u_{l,k_B,\sigma}(x, \sigma') \tag{6}$$

where $u_{l,k_B,\sigma}(x, \sigma')$ is the Bloch amplitude and L_0 is the system size. Moreover, we can easily establish the relation

$$u_{l,k_B,\sigma}(x, \sigma') = \delta_{\sigma',\sigma} u_{l,k_B}(x). \tag{7}$$

Note that $u_{l,k_B}(x)$ is also a periodic function and satisfies

$$u_{l,k_B}(x) = u_{l,k_B}(x + L). \tag{8}$$

In conclusion, the spinorial structure of the Bloch theory is trivial in the present case. This suggests to see what happens if some interaction in terms of spin are included. This issue has been addressed for a system in two dimensions and submitted to a confining potential [15]. This work will be generalized to get other interesting results and offer different discussions.

Before proceeding our generalization by including the magnetic field B, it is relevant to fix the starting point that is $B = 0$. This will allow us to underline the difference between these two cases and therefore make some comments. In fact, a Bloch theory in the two-dimensional periodic crystal within the spin–orbit coupling described by the Hamiltonian

$$H_0 = \frac{\hbar^2\vec{k}^2}{2m} - \frac{\hbar^2 k_{so}}{m}(\sigma_x k_z - \sigma_z k_x) + U(x) + V(z) \tag{9}$$

was considered in [15] where the second term is the Rashba interaction and the confining potential is fixed to be

$$V(z) = \frac{m\omega_0^2}{2}z^2. \tag{10}$$

Some interesting results have been derived and in particular the polarization symmetries have been discussed. In fact, we will show that these results can be recovered as a particular case of the present proposal when B is strong and then it can be used to control the polarization behaviors.

3. Gauge field coupling

To deal with different issues we introduce an external magnetic field B to end up with a confining potential in a similar way to (10) but with a B-dependent frequency. This will allow us to obtain an appropriate Hamiltonian that will be used to perform our task and in particular to generalize the result obtained in [15] to the case of B and offer some comments.

Let us start by considering a periodic crystal parameterized by (x, z) in the presence of a perpendicular B. The Hamiltonian for a single particle can be written as

$$H = \frac{\vec{\pi}^2}{2m} - \frac{\hbar^2 k_{so}}{m}(\sigma_x \pi_z - \sigma_z \pi_x) + U(x) \tag{11}$$

where the conjugate momentum is

$$\vec{\pi} = \vec{p} - \frac{e}{c}\vec{A}. \tag{12}$$

Unlike (9) the Rashba term is actually B-dependent and there is no external confinement analog to the potential $V(z)$. However, we will see that an analog form to $V(z)$ can spontaneously be created thanks to the emergence of B. Thus, one can expect that this effect will make a difference with respect to the results obtained in [15]. The corresponding quasi-momentum becomes

$$\vec{K} = \vec{k} - \frac{e}{\hbar c}\vec{A} \tag{13}$$

where $\vec{A}$ is the vector potential. These materials will be the starting point to establish a tool in order to tackle different problems in this work.

In this level, it is necessary to fix some quantities involved in the above Hamiltonian. In doing so, we choose the Landau gauge

$$\vec{A} = B(z, 0). \tag{14}$$

This allows (11) to take another form, such as

$$H = \frac{1}{2m}\left[p_z^2 + \left(p_x - \frac{eB}{c}z\right)^2\right] - \frac{\hbar^2 k_{so}}{m}\left[\sigma_x p_z - \sigma_z\left(p_x - \frac{eB}{c}z\right)\right] + U(x). \tag{15}$$

One can use different methods to solve this problem. In our analysis, we propose to split H into two parts and each one will be treated separately. They are

$$H = H^{(1)} + H^{(2)} \tag{16}$$

where the Hamiltonian $H^{(1)}$ is given by

$$H^{(1)} = \frac{\hbar^2}{2m}(k_x + \sigma_z k_{so})^2 + U(x) - \frac{\hbar^2 k_{so}^2}{2m} + \frac{\hbar^2 k_z^2}{2m} + \frac{1}{2m}\left(\frac{eB}{c}\right)^2 z^2 - \frac{\hbar e B}{mc}(k_x + \sigma_z k_{so})z \tag{17}$$

and $H^{(2)}$ is fixed to be

$$H^{(2)} = -\frac{\hbar^2 k_{so}}{m}\sigma_x k_z. \tag{18}$$

Clearly, due to the presence of B, the Bloch spinors will be changed from $|l, k_B, \sigma\rangle$ to the states $|l, k_B, \eta\rangle$. In fact, they verify the eigenvalue equation

$$H|l, k_B, \eta\rangle = \varepsilon_{l,\eta}(k_B)|l, k_B, \eta\rangle \tag{19}$$

where the parameter η will be fixed later. Then, the next step is to solve such equations in order to derive the whole spectrum that can be used to deal with different issues and in particular to discuss the symmetries.

4. Eigenvalue problems

As it will soon be clear, the above splitting is useful in the sense that the whole spectrum can be obtained in the easiest way. More precisely, we resort to a different spectrum entering in the game by an immediate derivation from the Bloch theory and making an algebraic approach to overcome some difficulties. In doing so, we split $H^{(1)}$ into two forms as well and analyze each part to get its spectrum. As far as the whole spectrum is concerned, we restrict ourselves to two first modes which allow us to make comparison with the results obtained in [15]. On the other hand, this restriction is argued by dealing with the Rashba interaction effect on the band structures [9].

4.1. $H^{(1)}$ spectrum

By separating the Hamiltonian $H^{(1)}$ into two parts, we show that the first spectrum can be easily obtained from that corresponding to the zero magnetic field case seen before. However, the second one can be deduced from a one-dimensional harmonic oscillator. This can be done by distinguishing two different cases: spin down and up. But, it is enough to handle only one of them and the other can be obtained in a similar way to reach the conclusion.

The simplest way to derive the corresponding spectrum is to divide $H^{(1)}$ itself into parts. The advantage of this is to refer to each part independently and also to make different comparisons with other cases. Then, the first one reads as

$$H_x^{(1)} = \frac{\hbar^2}{2m}(k_x + \sigma_z k_{so})^2 + U(x) - \frac{\hbar^2 k_{so}^2}{2m} \tag{20}$$

and the second is given by

$$H_z^{(1)} = \frac{\hbar^2 k_z^2}{2m} + \frac{1}{2m}\left(\frac{eB}{c}\right)^2 z^2 - \hbar\omega_c(k_x + \sigma_z k_{so})z \tag{21}$$

where $\omega_c = \frac{eB}{mc}$ is the cyclotron frequency. It is now clear that the second term in (21) plies the role of the confining potential (10). This will allow us to make contact with the one-dimensional harmonic oscillator in order to get the spectrum of (21).

The $H_x^{(1)}$ form is easy to handle where its eigenvalues and eigenstates can be derived in a similar way to those obtained in section 2. In fact, one can find the spectrum

$$\varepsilon_{l,\sigma}^{(1)}(k_B)\big|_x = \varepsilon_l^{(0)}(k_B + \sigma k_{so}) - \frac{\hbar^2 k_{so}^2}{2m} \tag{22}$$

where $\sigma = \pm 1$ is the spin index. It can also be written as

$$\varepsilon_{l,\sigma}^{(1)}(k_B)\big|_x = \frac{\hbar^2}{2m}\left(k_B^2 + 2\sigma k_{so}\right). \tag{23}$$

Clearly in the absence of the Rashba coupling, i.e. $k_{so} = 0$, we recover $\varepsilon_l^{(0)}(k_B)$ that corresponds to spinless particle.

In deriving the $H_z^{(1)}$ spectrum, we use some changes to end up with the harmonic oscillator form that is easy to diagonalize. We proceed by distinguishing two cases: spin up and down, i.e. $\sigma = \pm 1$, where we treat only one of them and the second will follow immediately. Then, we start by defining new variables for $\sigma = 1$, such as

$$P = \frac{p_z}{\sqrt{2m\hbar\omega_c}}, \qquad Q = \sqrt{\frac{m\omega_c}{2\hbar}}\, z. \tag{24}$$

They verify the commutation relation

$$[Q, P] = \frac{\mathrm{i}}{2}. \tag{25}$$

In terms of P and Q, $H_z^{(1)}\big|_{\sigma=1}$ takes the form

$$H_z^{(1)}\big|_{\sigma=1} = \hbar\omega_c\left[P^2 + Q^2 - \sqrt{\frac{2\hbar}{m\omega_c}}(k_B + k_{so})Q\right] \tag{26}$$

which is sharing some common features with the harmonic oscillator. To clarify this point, let us make an algebraic analysis by introducing the annihilation and creation operators. They can be realized as

$$a = Q + \mathrm{i}P + \lambda_{k,+1}, \qquad a^\dagger = Q - \mathrm{i}P + \lambda_{k,+1} \tag{27}$$

where the quantity $\lambda_{k,+1}$ is a function of the Bloch and rashba vectors, such as

$$\lambda_{k,+1} = -\sqrt{\frac{\hbar}{2m\omega_c}}(k_B + k_{so}). \tag{28}$$

It is easy to check the relation

$$[a, a^\dagger] = \mathbb{I}. \tag{29}$$

With these operators, $H_z^{(1)}\big|_{\sigma=1}$ can be mapped as

$$H_z^{(1)}\big|_{\sigma=1} = \hbar\omega_c\left[a^\dagger a + \tfrac{1}{2}\left(1 - 2\lambda_{k,+1}^2\right)\right]. \tag{30}$$

To make contact with the harmonic oscillator spectrum, first we define a new frequency in terms of ω_c and $\lambda_{k,+1}$. This is

$$\omega_{+1} = \omega_c\left(1 - 2\lambda_{k,+1}^2\right) \tag{31}$$

which allows us to write (30) as

$$H_z^{(1)}\big|_{\sigma=1} = \hbar\omega_{+1}\left(\frac{a^\dagger a}{1-2\lambda_{k,+1}^2} + \frac{1}{2}\right). \tag{32}$$

Note that hereafter we assume that the condition $\lambda_{k,+1} \neq \pm\sqrt{\frac{1}{2}}$ is fulfilled. In fact, we return to analyze this critical point separately and underline its influences on the system behavior. Secondly, we can consider new operators in terms of a and $a^\dagger$ involved before. They are

$$b = \frac{1}{\sqrt{1-2\lambda_{k,+1}^2}}a, \qquad b^\dagger = \frac{1}{\sqrt{1-2\lambda_{k,+1}^2}}a^\dagger. \tag{33}$$

They lead to the Hamiltonian

$$H_z^{(1)}\big|_{\sigma=1} = \hbar\omega_{+1}\left(b^\dagger b + \tfrac{1}{2}\right). \tag{34}$$

This is nothing but the harmonic oscillator Hamiltonian and therefore the spectrum can easily be obtained. Indeed, the eigenvalues are given by

$$\varepsilon_{l,n,+1}^{(1)}(k_B)\big|_z = \hbar\omega_{+1}\left(n + \tfrac{1}{2}\right), \qquad n = 0, 1, 2\ldots \tag{35}$$

and the eigenstates read as

$$|n\rangle = \frac{(b^\dagger)^n}{\sqrt{n!}}|0\rangle. \tag{36}$$

To complete the above analysis we need to consider the second case where $\sigma = -1$. Indeed, the same calculations remain valid and the only thing that should be changed is $\lambda_{k,+1}$. Its analog can be defined in a similar way, such as

$$\lambda_{k,-1} = -\sqrt{\frac{\hbar}{2m\omega_c}}(k_B - k_{so}). \tag{37}$$

It is easy to observe $\lambda_{k,+1}$ and $\lambda_{k,-1}$ verify the symmetry relation

$$\lambda_{k,+1}(k_B) = -\lambda_{k,-1}(-k_B). \tag{38}$$

Therefore, from the above analysis we end up with the eigenvalues for the spin down case, such as

$$\varepsilon_{l,n,-1}^{(1)}(k_B)\big|_z = \hbar\omega_{-1}\left(n + \tfrac{1}{2}\right), \qquad n = 0, 1, 2\ldots \tag{39}$$

where ω_{+1} and ω_{-1} satisfy a similar relation to (38). These last spectrums will be used to analyze the critical point. This will allow us to make contact with the Dirac fermions in the presence of the magnetic field.

At this level, we have all ingredients to get the $H^{(1)}$ spectrum. Indeed, we just combine all together to obtain those that verify the eigenvalue equation

$$H^{(1)}|l, k_B, n, \sigma\rangle = \varepsilon_{l,\sigma}^{(1)}(k_B)|l, k_B, n, \sigma\rangle \tag{40}$$

which shows that the eigenvalues of $H^{(1)}$ are of the form

$$\varepsilon_{l,n,\sigma}^{(1)}(k_B) = \varepsilon_l^{(0)}(k_B + \sigma k_{so}) - \frac{\hbar^2 k_{so}^2}{2m} + \hbar\omega_\sigma\left(n + \tfrac{1}{2}\right). \tag{41}$$

It is clear that the corresponding eigenstates can be obtained as a tensor product, namely

$$|l, k_B, n, \sigma\rangle = |l, k_B, \sigma\rangle \otimes |n\rangle. \tag{42}$$

The $H^{(1)}$ eigenfunctions can be obtained by projecting the above states on the coordinate representation. Thus, they can be found in terms of the Bloch amplitude as

$$\langle x, n', \sigma'|l, k_B, n, \sigma\rangle = \frac{1}{\sqrt{L_0}}\delta_{n',n}\delta_{\sigma',\sigma}\,\mathrm{e}^{\mathrm{i}k_B x}u_{l,k_B+\sigma k_{so}}(x). \tag{43}$$

With this we complete the derivation of the eigenvalues and the eigenstates problems of $H^{(1)}$. These will be used to derive other quantities and discuss different issues.

4.2. Total spectrum

After getting the $H^{(1)}$ spectrum we need to determine that corresponding to $H^{(2)}$ in order to end up with the total eigenvalues and eigenstates. In this case, the Bloch amplitude acquires a new spinoral structure, such as

$$\langle x, n, \sigma | l, k_B, \eta\rangle = \frac{1}{\sqrt{L_0}} \mathrm{e}^{\mathrm{i}k_B x} u_{l,k_B,\eta}(x, n, \sigma) \tag{44}$$

where $u_{l,k_B,\eta}(x, n, \sigma)$ is also a periodic function, namely

$$u_{l,k_B,\eta}(x, n, \sigma) = u_{l,k_B,\eta}(x + L, n, \sigma). \tag{45}$$

Before proceeding, let us denote by $\theta_{l,k_B,\eta}(n, \sigma)$ the Bloch spinors in the $\{l, k_B, n, \sigma\}$ representation, where η will be fixed later. They can be written as

$$\theta_{l,k_B,\eta}(n, \sigma) = \langle l, k_B, n, \sigma | l, k_B, \eta\rangle \tag{46}$$

which allow us to derive the relation

$$\langle l', k'_B, n, \sigma | l, k_B, \eta\rangle = \delta_{l',l}\delta_{k'_B,k_B}\theta_{l,k_B,\eta}(n, \sigma). \tag{47}$$

To explicitly determine $\theta_{l,k_B,\eta}(n, \sigma)$, we have to solve the eigenvalue equation (19) where still only $H^{(2)}$ is to be treated. In doing so, let us project this equation on the states $|l, k_B, n, \sigma\rangle$ to get the Bloch spinors as

$$\begin{aligned}\sum_{n',\sigma'} [\langle l', k'_B, n, \sigma | H^{(1)} | l, k_B, n', \sigma'\rangle + \langle l', k'_B, n, \sigma | H^{(2)} | l, k_B, n', \sigma'\rangle]\theta_{l,k_B,\eta}(n', \sigma')\\ = \varepsilon_{l,\eta(k_B)}\delta_{l',l}\delta_{k'_B,k_B}\theta_{l,k_B,\eta}(n, \sigma).\end{aligned} \tag{48}$$

This is actually involving two kinds of matrix elements where each one can separately be evaluated because they are related to $H^{(1)}$ and $H^{(2)}$. Indeed, for the first it is easy to obtain

$$\langle l', k'_B, n, \sigma | H^{(1)} | l, k_B, n', \sigma'\rangle = \varepsilon^{(1)}_{l,n,\sigma}(k_B)\delta_{l',l}\delta_{k'_B,k_B}\delta_{\sigma,\sigma'}. \tag{49}$$

The second kind can be written as

$$\langle l', k'_B, n, \sigma | H^{(2)} | l, k_B, n', \sigma'\rangle = -\frac{\hbar^2 k_{so}}{m}\langle\sigma|\sigma_x|\sigma'\rangle\langle n|k_z|n'\rangle\delta_{l',l}\delta_{k'_B,k_B}. \tag{50}$$

This also contains two separated matrix elements. Concerning the first one, it is easy to end up with the result

$$\langle\sigma|\sigma_x|\sigma'\rangle = 1 - \delta_{\sigma,\sigma'}. \tag{51}$$

Combining all together to write the final equation for the eigenvalues $\varepsilon_{l,\eta}(k_B)$ and eigenspinors $\theta_{l,k_B,\eta}(n, \sigma)$ as

$$\begin{aligned}\sum_{n',\sigma'} \left[\varepsilon^{(1)}_{l,n,\sigma}(k_B)\delta_{l',l}\delta_{k'_B,k_B}\delta_{\sigma,\sigma'} - \frac{\hbar^2 k_{so}}{m}(1 - \delta_{\sigma,\sigma'})\langle n|k_z|n'\rangle\delta_{l',l}\delta_{k'_B,k_B}\right]\theta_{l,k_B,\eta}(n', \sigma')\\ = \varepsilon_{l,\eta(k_B)}\delta_{l',l}\delta_{k'_B,k_B}\theta_{l,k_B,\eta}(n, \sigma).\end{aligned} \tag{52}$$

Actually, this depends on different quantum numbers and can be solved by making use of some restrictions. This will be the subject of the remaining subsection.

To solve the above equation, we may focus on particular cases as these allow us to simplify different quantities. Indeed, assuming that we have the same band indices $l' = l$ and Bloch's quasi-momentum $k_B = k'_B$, then (52) can be reduced to

$$\sum_{n',\sigma'} \left\{\delta_{n,n'}\delta_{\sigma,\sigma'}\varepsilon^{(1)}_{l,n,\sigma}(k_B) - \frac{\hbar^2 k_{so}}{2m}(1 - \delta_{\sigma,\sigma'})\langle n|k_z|n'\rangle\right\}\theta_{l,k_B,\eta}(n', \sigma') = \varepsilon_{l,\eta}(k_B)\theta_{l,k_B,\eta}(n, \sigma). \tag{53}$$

To go further, we should evaluate the second term in (53). This can be done by returning to the annihilation and creation operators in order to obtain

$$A - A^{\dagger} = \frac{2\mathrm{i}\hbar}{\sqrt{1 - 2\lambda_{k,\sigma}^2}} P. \tag{54}$$

This is leading to write k_z as

$$k_z = \mathrm{i}(A - A^{\dagger}) \sqrt{\frac{m\omega_c}{2\hbar}\left(\frac{1}{2} - \lambda_{k,\sigma}^2\right)}. \tag{55}$$

Now it is easy to get the matrix element of k_z as

$$\langle n|k_z|n'\rangle = \pm \mathrm{i}\delta_{n,n'\pm 1} \sqrt{\frac{m\omega_\sigma}{2\hbar}\left(n + \frac{1}{2} \mp \frac{1}{2}\right)}. \tag{56}$$

Therefore, the eigenvalue equation takes the form

$$\sum_{n',\sigma'} \left\{ \delta_{n,n'}\delta_{\sigma,\sigma'} \left[\frac{\hbar^2}{2m}\left(k_B^2 + 2\sigma k_{so}\right) + \hbar\omega_\sigma \left(n + \frac{1}{2}\right)\right] \mp \mathrm{i}\frac{\hbar^2 k_{so}}{2m}(1 - \delta_{\sigma,\sigma'})\delta_{n,n'\pm 1} \right.$$
$$\left. \times \sqrt{\frac{m\omega_\sigma}{2\hbar}\left(n + \frac{1}{2} \mp \frac{1}{2}\right)} \right\} \theta_{l,k_B,\eta}(n',\sigma') = \varepsilon_{l,\eta}(k_B)\theta_{l,k_B,\eta}(n,\sigma). \tag{57}$$

Many discussions can be reported on this spectrum as well as its corresponding eigenfunctions. In the following, we treat a special case that allows us to make contact with other results and give different comments. More precisely, since we are hoping to make comparison with the proposal developed in [15], we restrict ourselves only to the two first transverse modes, i.e. $n = 0, 1$. On the other hand, such approximation has been motivated by analyzing the spin accumulation in quantum wires with strong Rashba coupling [9]. This restriction will allow us to simply (57) and derive interesting results.

Focusing on $n = 0, 1$ and $\sigma = \pm 1$, (57) reduces to a diagonalization of the 4×4 matrix, which is a task that can be done analytically. This process leads to fix η as a number running on $(1, \ldots, 4)$ and to end up with two relations between different energies. They are

$$\varepsilon_{l,\eta=1,2}(k_B) = \varepsilon_l^+(k_B) - \xi_{l_{1,2}}(k_B), \qquad \varepsilon_{l,\eta=3,4}(k_B) = \varepsilon_l^+(k_B) + \xi_{l_{2,1}}(k_B) \tag{58}$$

where $\varepsilon_l^+(k_B)$ reads as

$$\varepsilon_l^+(k_B) = \frac{1}{2}\left[\varepsilon_l^{(0)}(k_B + k_{so}) + \varepsilon_l^{(0)}(k_B - k_{so})\right] + \hbar\omega_c\left(1 - 2\lambda_{k,\sigma}^2\right) - \frac{\hbar^2 k_{so}^2}{2m} \tag{59}$$

and the quantity $\xi_{l_{1,2}}(k_B)$ is given by

$$\xi_{l_{1,2}}(k_B) = \sqrt{\xi^2 + \left[\varepsilon_l^-(k_B) \mp \frac{\hbar\omega_c}{2}\left(1 - 2\lambda_{k,\sigma}^2\right)\right]^2}. \tag{60}$$

The energy $\varepsilon_l^-(k_B)$ is

$$\varepsilon_l^-(k_B) = \tfrac{1}{2}\left\{\varepsilon_l^{(0)}(k_B + k_{so}) - \varepsilon_l^{(0)}(k_B - k_{so})\right\} \tag{61}$$

and ξ takes the form

$$\xi = \frac{\hbar^2 k_{so}}{2}\sqrt{\frac{\omega_c}{m\hbar}\left(1 - 2\lambda_{k,\sigma}^2\right)}. \tag{62}$$

We show that there is a hidden symmetry that is encoded in the spectrum. Indeed, from the Bloch theory it is easy to verify the relation

$$\varepsilon_l^{(0)}(k_B) = \varepsilon_l^{(0)}(-k_B) \tag{63}$$

which allows us to show that the different energies are connected via

$$\varepsilon_{l,\eta=1}(k_B) = \varepsilon_{l,\eta=2}(-k_B), \qquad \varepsilon_{l,\eta=3}(k_B) = \varepsilon_{l,\eta=4}(-k_B). \tag{64}$$

The corresponding normalized eigenspinors $\theta_{l,k_B,\eta}(n,\sigma)$ can be obtained by returning to the eigenvalue equation. This leads to

$$\theta_{l,k_B,\eta=1,4} = N_{l,k_B,\eta=1,4}^{-\frac{1}{2}}\tilde{\theta}_{l,k_B,\eta=1,4}, \qquad \theta_{l,k_B,\eta=2,3} = N_{l,k_B,\eta=2,3}^{-\frac{1}{2}}\tilde{\theta}_{l,k_B,\eta=2,3} \tag{65}$$

where the functions $\theta_{l,k_B,\eta=1,4}$ take the form

$$\tilde{\theta}_{l,k_B,\eta=1,4} = \begin{pmatrix} \frac{\mathrm{i}}{\xi}\left[\varepsilon_l^-(k_B) - \frac{\hbar\omega_c}{2}\left(1-2\lambda_{k,\sigma}^2\right) \mp \xi_{l_1}(k_B)\right] \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{66}$$

and the remaining quantities read as

$$\tilde{\theta}_{l,k_B,\eta=2,3} = \begin{pmatrix} 0 \\ -\frac{\mathrm{i}}{\xi}\left[\varepsilon_l^-(k_B) + \frac{\hbar\omega_c}{2}\left(1-2\lambda_{k,\sigma}^2\right) \pm \xi_{l_2}(k_B)\right] \\ 1 \\ 0 \end{pmatrix}. \tag{67}$$

Note that we have involved such notation

$$\theta_{l,k_B,\eta} \equiv \begin{pmatrix} \theta_{l,k_B,\eta}(n=0,\sigma=+1) \\ \theta_{l,k_B,\eta}(n=0,\sigma=-1) \\ \theta_{l,k_B,\eta}(n=1,\sigma=+1) \\ \theta_{l,k_B,\eta}(n=1,\sigma=-1) \end{pmatrix} \tag{68}$$

in deriving the eigenspinors, the normalized constants can be obtained as usual. Thus, it is not hard to show that there are

$$N_{l,k_B,\eta} = \sum_{n=0}^{1}\sum_{\sigma=-1}^{+1} |\tilde{\theta}_{l,k_B,\eta}(n,\sigma)|^2. \tag{69}$$

Finally, using the obtained spectrum it is easy to verify the relation

$$N_{l,k_B,\eta=1,4} = N_{l,-k_B,\eta=2,3} \tag{70}$$

between different sub-bands entering in the game.

At this level, let us give a comment about the band structures. Indeed, calculating the derivative of $\varepsilon_l^-(k_B)$ with respect to k_B at the zero point, we obtain the group velocity

$$\left.\frac{d\varepsilon_l^-(k_B)}{dk_B}\right|_{k_B=0} = v_l^{(0)}(k_{so}) \tag{71}$$

which is nothing but that corresponding to the one-dimensional system. Clearly, starting from the obtained spectrum one can observe that the band splitting near $k_B = 0$ is a linear function in terms of the Bloch vector k_B. It is similar to what has experimentally been observed for two-dimensional electron gazes [21]. Our conclusion is in agreement with what has been reached in [15].

To close this section let us note that when the magnetic field is switched off, we recover Bloch Hamiltonian amended by a free particle living on the z-direction. To get the analysis of [15] we can simply add a harmonic potential in the same direction. Our study shows that this potential can simply be generated from the confinement due to the presence of B. On the other hand, we will use the established tools to discuss the symmetry in terms of the polarizations.

5. Polarization symmetries

We start by noting that it has been shown the system described by the Hamiltonian (9) is preserving the polarization symmetries [15]. Thus, it is natural to ask about such symmetries for the present system in order to underline the similarities and differences. To reply to this question, we start discussing the polarizations by adopting the definition

$$P^{(i)}_{l,\eta}(k_B) \equiv \langle l, k_B, \eta|\sigma_i|l, k_B, \eta\rangle \tag{72}$$

where σ_i are the Pauli spin operators, with $i = x, y, z$ and $\eta = 1, 2, 3, 4$. Therefore, we need to evaluate different quantities entering in the game.

We start by analyzing each component separately. In doing so, we write the identity operator in the $\{|l, k_B, n, \sigma\rangle\}$ basis and take into account the structure of the obtained Bloch spinors. This process leads to show that its expression in the x-direction is given by

$$P^{(x)}_{l,\eta} = \sum_{n=0}^{1} \sum_{\sigma',\sigma''=-1}^{+1} \theta^{*}_{l,k_B,\eta}(n, \sigma')(1 - \delta_{\sigma',\sigma''})\theta_{l,k_B,\eta}(n, \sigma'') = 0 \tag{73}$$

as well as for the y-direction

$$P^{(y)}_{l,\eta} = \sum_{n=0}^{1} \sum_{\sigma',\sigma''=-1}^{+1} \theta^{*}_{l,k_B,\eta}(n, \sigma')i^{\sigma''}(1 - \delta_{\sigma',\sigma''})\theta_{l,k_B,\eta}(n, \sigma'') = 0 \tag{74}$$

for all l and k_B in the Brillouin zone. Clearly, both polarizations longitudinal and perpendicular are nulls, which are in agreement with the analysis of [15].

Let us now analyze the polarization along the z-direction. Indeed, for such cases we show that (72) reduces to the quantity

$$P^{(z)}_{l,\eta} = \sum_{n=0}^{1} \sum_{\sigma=-1}^{+1} \theta^{*}_{l,k_B,\eta}(n, \sigma)\sigma\theta_{l,k_B,\eta}(n, \sigma). \tag{75}$$

To explicitly evaluate this form, we need to separate the two cases governed by the four Bloch sub-bands and use of course the obtained spinors. Thus, for $\eta = 1, 4$ we find

$$P^{(z)}_{l,\eta} = N^{-1}_{l,k_B,\eta=1,4}\left\{\frac{1}{\xi^2}\left[\varepsilon^{-}_{l}(k_B) - \frac{\hbar\omega_c}{2}\left(1 - 2\lambda^2_k\right) \pm \xi_{l_1}(k_B)\right]^2 - 1\right\} \tag{76}$$

and for $\eta = 2, 3$ we end up with

$$P^{(z)}_{l,\eta} = N^{-1}_{l,k_B,\eta=2,3}\left\{1 - \frac{1}{\xi^2}\left[\varepsilon^{-}_{l}(k_B) + \frac{\hbar\omega_c}{2}\left(1 - 2\lambda^2_k\right) \mp \xi_{l_2}(k_B)\right]^2\right\}. \tag{77}$$

This shows that this component of polarizations has a finite value, which makes difference with respect to those seen above. Moreover, using (70) and the relations

$$\varepsilon^{-}_{l}(k_B) = -\varepsilon^{-}_{l}(-k_B), \qquad \xi_{l_{1,2}}(k_B) \neq \xi_{l_{2,1}}(-k_B) \tag{78}$$

we show that the symmetry relation for the polarizations verifies

$$P^{(z)}_{l,\eta=1,4}(k_B) \neq -P^{(z)}_{l,\eta=2,3}(-k_B). \tag{79}$$

This tells us that the polarization symmetries are not preserved for the present system. These have been broken by the external magnetic field that interact with the particles in terms of the gauge field. Its preservation can be recovered by switching off B and adding a confining potential along the z-direction [15]. However, as we will see in the following, we can have such preservation of the symmetries by considering a limit of strong B.

6. Dynamical spin analysis

Firstly, let us note that the present system is sharing some common features with that describing the spin Hall effect (SHE). In fact, to talk about SHE one has to consider the Rashba or Dressellhaus spin–orbit coupling as an important ingredient in the Hamiltonian system. This suggests to make a dynamical spin analysis of our system in order to underline its behavior. This can be done by determining the velocity and spin current operators, which can be used to write down the corresponding spin Hall conductivity $\sigma_{\mathrm{H}}^{\mathrm{s}}$.

To do our task we start by evaluating the velocity operators in different directions. Indeed, from the Hamiltonian formalism one can easily obtain the quantities

$$v_x = \frac{\pi_x}{m} + \frac{\hbar k_{so}}{m}\sigma_z, \qquad v_z = \frac{\pi_z}{m} + \frac{\hbar k_{so}}{m}\sigma_x \tag{80}$$

which will be used to derive the spin current operator. To do this, let us recall that due to the fact that spin is not a conserved quantity in the presence of the spin–orbit coupling, there are different definitions of such operators [16]. But we adopt that this is widely used where the spin current operator is defined as the difference of the conserved densities for carriers with opposite spins. This is

$$J_z^{\sigma_y} = \frac{\hbar}{4}(v_z\sigma_y + \sigma_y v_z) \tag{81}$$

leading to the form

$$J_z^{\sigma_y} = \frac{\hbar\pi_z}{2m}\sigma_y. \tag{82}$$

Note that $J_z^{\sigma_y}$ can also be obtained by using another method. The advantage of this latter is to end up with an appropriate form for $\sigma_{\mathrm{H}}^{\mathrm{s}}$, which we will do soon.

Recall that SHE is a manifestation of a collective motion of the spins of particles in an external electric field. This phenomenon is characterized by $\sigma_{\mathrm{H}}^{\mathrm{s}}$ which is related to that for the quantum Hall effect [17]. From a linear response formalism, it can be written in terms of the frequency ω as [16]

$$\sigma_{\mathrm{H}}^{\mathrm{S}}(\omega) = -\frac{e\hbar}{iL^2}\sum_{\alpha,\alpha'}\frac{\langle\alpha|J_y^{\sigma_z}|\alpha'\rangle\langle\alpha'|v_x|\alpha'\rangle}{\epsilon_\alpha - \epsilon_{\alpha'} + \hbar\omega + i0}\,\frac{f(\epsilon_\alpha) - f(\epsilon_{\alpha'})}{\epsilon_\alpha - \epsilon_{\alpha'}} \tag{83}$$

where we have set $|\alpha\rangle \equiv |l, k_B, n, \sigma\rangle$ and $\epsilon_\alpha \equiv \varepsilon_{l,\eta}(k_B)$. It can be simplified by introducing $J_y^{\sigma_z}$ in terms of the Heisenberg equation, namely

$$J_z^{\sigma_y} = -\frac{\mathrm{i}}{4}[H, \sigma_z]. \tag{84}$$

The corresponding matrix element in the basis $|\alpha\rangle$ can be evaluated as

$$\langle\alpha|J_z^{\sigma_y}|\alpha'\rangle = -\frac{\mathrm{i}}{4}(\epsilon_\alpha - \epsilon_{\alpha'})\langle\alpha|\sigma_z|\alpha'\rangle. \tag{85}$$

Injecting this in (83), we end up with

$$\sigma_{\mathrm{H}}^{\mathrm{S}}(\omega) = \frac{e\hbar}{4L^2}\sum_{\alpha,\alpha'}\frac{\langle\alpha|\sigma_z|\alpha'\rangle\langle\alpha'|v_x|\alpha'\rangle}{\epsilon_\alpha - \epsilon_{\alpha'} + \hbar\omega + i0} f(\epsilon_\alpha) - f(\epsilon_{\alpha'}). \tag{86}$$

To explicitly determine such conductivity one may use the self-consistent Born approximation in a similar way to [16]. On the other hand, (86) can be analyzed by adopting the high and low frequency regimes to reach different conclusions.

7. Analysis of the two cases

We are hoping to analyze different limits in order to characterize the system behavior. For this section we start by considering the cases where the Rashba coupling constant and the periodic potential are nulls. These will allow us to end up with a Landau problem amended either with a periodic potential or the Rashba term in the presence of the magnetic field.

7.1. Without Rashba coupling

The first one concerns the Rashba coupling constant when it is absent. In fact, we return back to the former studies and show its restrictions to such cases. In doing so, we derive the corresponding spectrum as well as discuss the polarizations symmetries. Before doing our job, let us fix the Hamiltonian that describes the present limit. This can easily be obtained from (15) to get

$$H|_{k_{so}=0} = \frac{1}{2m}\left[p_z^2 + \left(p_x - \frac{eB}{c}z\right)^2\right] + U(x) \tag{87}$$

which is nothing but the Landau problem written in the Landau gage and submitted to the constraint of a periodic potential. Clearly, without B we have a Bloch theory along the x-direction plus a wave plane along the z-direction.

To get the eigenvalues and the eigenstates, we algebraically diagonalize the above Hamiltonian. To proceed, let us define the annihilation and creation operators as

$$c = \frac{l_B}{\sqrt{2}\hbar}\left[p_z + \mathrm{i}\left(p_x - \frac{eB}{c}z\right)\right], \qquad c^\dagger = \frac{l_B}{\sqrt{2}\hbar}\left[p_z + \mathrm{i}\left(p_x - \frac{eB}{c}z\right)\right] \tag{88}$$

where $l_B = \sqrt{\frac{c\hbar}{eB}}$ is the magnetic length. It is easy to verify the commutation relation

$$[c, c^\dagger] = \mathbb{I}. \tag{89}$$

In terms of these operators, (87) can be written as

$$H|_{k_{so}=0} = \hbar\omega_c\left(c^\dagger c + \tfrac{1}{2}\right) + U(x). \tag{90}$$

Therefore the corresponding spectrum is given by

$$E_n|_{k_{so}=0}(k_B) = \hbar\omega_c\left(n + \tfrac{1}{2}\right), \qquad |n, k_B\rangle, \qquad n = 0, 1, 2, \ldots \tag{91}$$

where the states explicitly read as

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle, \qquad \langle x|k_B\rangle = \frac{1}{\sqrt{L_0}}\,\mathrm{e}^{\mathrm{i}k_B x}u_{k_B}(x). \tag{92}$$

To close this part, we note that the eigenvalues depend only on n and are completely independent of the Bloch momentum $\hbar k_B$. Due to the lack of dependence of the energy on k_B, the degeneracy of each level is enormous.

7.2. Without periodic potential

Let us study another case that corresponds to $U(x)$ equal to zero. This will allows us to characterize the nature of the present system when $U(x)$ is switched off. In doing so, we give the corresponding Hamiltonian, which can be deduced from (15) as

$$H|_{U(x)=0} = \frac{1}{2m}\left(\pi_z^2 + \pi_x^2\right) - \frac{\hbar^2 k_{so}}{m}(\sigma_x\pi_z - \sigma_z\pi_x). \tag{93}$$

Before proceeding we note that an analog operator has been considered in different contexts but when only the x and y coordinates are involved, for instance in [18]. Here it is different because of the last term which includes diagonal and off-diagonal matrices.

To get the spectrum of (93) we adopt an analysis similar to that reported in subsection (4.1) except that we take $U(x) = 0$. We also note that for each state we can still choose k_x freely, thus we have degenerate energy levels. If we assume our space is finite, for definiteness we choose a rectangle of dimensions $L_x \times L_y$, the number of states in a level is finite and can be calculated. The eigenvalues of p_x are quantized as $k_x = \frac{2\pi}{L_y} j$ where j is a quantum number. Therefore, after diagonalizing (93) we end up with the relations

$$\varepsilon_{l,\eta=1,2}(j) = \varepsilon_l^+(j) - \xi_{l_{1,2}}(j), \qquad \varepsilon_{l,\eta=3,4}(j) = \varepsilon_l^+(j) + \xi_{l_{2,1}}(j) \tag{94}$$

where $\varepsilon_l^+(j)$ takes the form

$$\varepsilon_l^+(j) = \frac{\hbar^2}{2m}\left(\frac{2\pi j}{L_y}\right)^2 + \hbar\omega_c\left(1 - 2\lambda_{j,\sigma}^2\right) \tag{95}$$

and $\xi_{l_{1,2}}(j)$ reads as

$$\xi_{l_{1,2}}(j) = \sqrt{\xi^2 + \left[\varepsilon_l^-(j) \mp \frac{\hbar\omega_c}{2}\left(1 - 2\lambda_{j,\sigma}^2\right)\right]^2}. \tag{96}$$

The energy $\varepsilon_l^-(j)$ is given by

$$\varepsilon_l^-(j) = \frac{\hbar^2}{m}\left(\frac{2\pi j}{L_y}\right) k_{so} \tag{97}$$

and the quantity ξ is

$$\xi = \frac{\hbar^2 k_{so}}{2}\sqrt{\frac{\omega_c}{m\hbar}\left(1 - 2\lambda_{j,\sigma}^2\right)} \tag{98}$$

where the parameter $\lambda_{j,\sigma}$ becomes

$$\lambda_{j,\sigma} = -\sqrt{\frac{\hbar}{2m\omega_c}}\left(\frac{2\pi j}{L_y} + \sigma k_{so}\right). \tag{99}$$

These lead to derive the normalized eigenspinors $\theta_{l,j,\eta}(n,\sigma)$ as

$$\theta_{l,j,\eta=1,4} = N_{l,j,\eta=1,4}^{-\frac{1}{2}}\tilde{\theta}_{l,j,\eta=1,4}, \qquad \theta_{l,k_B,\eta=2,3} = N_{l,j,\eta=2,3}^{-\frac{1}{2}}\tilde{\theta}_{l,j,\eta=2,3} \tag{100}$$

where the spinors $\theta_{l,j,\eta=1,4}$ read as

$$\tilde{\theta}_{l,j,\eta=1,4} = \begin{pmatrix} \frac{\mathrm{i}}{\xi}\left[\varepsilon_l^-(j) - \frac{\hbar\omega_c}{2}\left(1 - 2\lambda_{j,\sigma}^2\right) \mp \xi_{l_1}(j)\right] \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{101}$$

and $\theta_{l,j,\eta=2,3}$ are given by

$$\tilde{\theta}_{l,j,\eta=2,3} = \begin{pmatrix} 0 \\ -\frac{\mathrm{i}}{\xi}\left[\varepsilon_l^-(j) + \frac{\hbar\omega_c}{2}\left(1 - 2\lambda_{j,\sigma}^2\right) \pm \xi_{l_2}(j)\right] \\ 1 \\ 0 \end{pmatrix}. \tag{102}$$

It is natural to ask about the polarization symmetries in the present case. The answer can be obtained by evaluating different components and to obtain for $\eta = 1, 4$ the result

$$P_{l,\eta}^{(z)} = N_{l,j,\eta=1,4}^{-1}\left\{\frac{1}{\xi^2}\left[\varepsilon_l^-(j) - \frac{\hbar\omega_c}{2}\big(1-2\lambda_{j,\sigma}^2\big) \pm \xi_{l_1}(j)\right]^2 - 1\right\} \tag{103}$$

and for $\eta = 2, 3$ we have

$$P_{l,\eta}^{(z)} = N_{l,j,\eta=2,3}^{-1}\left\{1 - \frac{1}{\xi^2}\left[\varepsilon_l^-(j) + \frac{\hbar\omega_c}{2}\big(1-2\lambda_{j,\sigma}^2\big) \mp \xi_{l_2}(j)\right]^2\right\}. \tag{104}$$

By showing the relation

$$P_{l,\eta=1,4}^{(z)}(j)|_{U(x)=0} \neq -P_{l,\eta=2,3}^{(z)}(-j)|_{U(x)=0}. \tag{105}$$

we conclude that such symmetries are also not preserved in the present case. This closes the analysis of the Rashba and periodic potential terms. In the following, we will study other cases.

8. Strong magnetic field limit

It is interesting to analyze the case where the champ magnetic is strong. Such interest comes from the fact that this limit is important for the quantum Hall effect. Thus, it will be a good task to deal with such a limit and derive different results.

Recall that B is included in the Hamiltonian for the z-direction. Requiring that B is strong, we can approximate (21) as

$$H_z^{(1)}\big|_{\mathrm{sB}} = \frac{\hbar^2 k_z^2}{2m} + \frac{1}{2m}\left(\frac{eB}{c}\right)^2 z^2 \tag{106}$$

which is nothing but the one-dimensional harmonic oscillator of frequency ω_c, where the abbreviation (sB) denotes strong magnetic field. Therefore, the corresponding spectrum is given by

$$E_n|_{\mathrm{sB}} = \hbar\omega_c\big(n+\tfrac{1}{2}\big). \tag{107}$$

This modifies the $H^{(1)}$ eigenvalues as

$$\varepsilon_{l,n,\sigma}^{(1)}(k_B)\big|_{\mathrm{sB}} = \varepsilon_l^{(0)}(k_B + \sigma k_{so}) - \frac{\hbar^2 k_{so}^2}{2m} + \hbar\omega_c\big(n+\tfrac{1}{2}\big) \tag{108}$$

as well as the matrix element

$$\langle n|k_z|n'\rangle = \pm \mathrm{i}\delta_{n,n'\pm 1}\sqrt{\frac{m\omega_c}{2\hbar}\left(n+\frac{1}{2}\mp\frac{1}{2}\right)}. \tag{109}$$

Note that what makes the difference with respect to the previous analysis is that the parameter $\lambda_{k,\sigma}$ is absent. This of course will offer different simplifications and determine interesting results concerning the polarizations.

To be coherent with our analysis, we consider the two first transverses modes and derive the whole spectrum. Doing this to get

$$\varepsilon_{l,\eta=1,2}(k_B)\big|_{\mathrm{sB}} = \varepsilon_l^+(k_B) - \xi_{l_{1,2}}(k_B)\big|_{\mathrm{sB}}, \qquad \varepsilon_{l,\eta=3,4}(k_B)\big|_{\mathrm{sB}} = \varepsilon_l^+(k_B) + \xi_{l_{2,1}}(k_B)\big|_{\mathrm{sB}} \tag{110}$$

where different quantities changed to

$$\varepsilon_l^+(k_B)|_{\mathrm{sB}} = \frac{1}{2}\big\{\varepsilon_l^{(0)}(k_B+k_{so}) + \varepsilon_l^{(0)}(k_B-k_{so})\big\} + \hbar\omega_c - \frac{\hbar^2 k_{so}^2}{2m} \tag{111}$$

as well as

$$\xi_{l_{1,2}}(k_B)|_{\rm sB} = \sqrt{\xi^2|{\rm sB} + \left(\varepsilon_l^-(k_B) \mp \frac{\hbar\omega_c}{2}\right)^2} \tag{112}$$

and we have set

$$\xi|_{\rm sB} = \frac{\hbar^2 k_{so}}{m}\sqrt{\frac{m\omega_c}{2\hbar}}. \tag{113}$$

The corresponding Bloch spinors are given by

$$\tilde{\theta}_{l,k_B,\eta=1,4}|_{\rm sB} = \begin{pmatrix} \frac{\rm i}{\xi|_{\rm sB}}\left[\varepsilon_l^-(k_B) - \frac{\hbar\omega_c}{2} \mp \xi_{l_1}(k_B)|_{\rm sB}\right] \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{114}$$

and the other sub-band reads as

$$\tilde{\theta}_{l,k_B,\eta=2,3}|_{\rm sB} = \begin{pmatrix} 0 \\ -\frac{\rm i}{\xi|_{\rm sB}}\left[\varepsilon_l^-(k_B) + \frac{\hbar\omega_c}{2} \pm \xi_{l_2}(k_B)|_{\rm sB}\right] \\ 1 \\ 0 \end{pmatrix}. \tag{115}$$

It is relevant to discuss the polarization symmetries in the present case to underline what makes difference with the previous analysis. On the other hand, the above analysis is sharing some common features with that realized in [15]. Indeed, using the above tools to show that for $\eta = 1, 4$ we have

$$P_{l,\eta}^{(z)}|_{\rm sB} = N_{l,k_B,\eta=1,4}^{-1} \times \left\{\frac{1}{\xi^2|_{\rm sB}}\left[\varepsilon_l^-(k_B) - \frac{\hbar\omega_c}{2} \mp \xi_{l_1}(k_B)|_{\rm sB}\right]^2 - 1\right\} \tag{116}$$

as well as for $\eta = 2, 3$

$$P_{l,\eta}^{(z)}|_{\rm sB} = N_{l,k_B,\eta=2,3}^{-1} \times \left\{1 - \frac{1}{\xi^2|_{\rm sB}}\left[\varepsilon_l^-(k_B) + \frac{\hbar\omega_c}{2} \pm \xi_{l_2}(k_B)|_{\rm sB}\right]^2\right\}. \tag{117}$$

Finally, we end up with

$$P_{l,\eta=1,4}^{(z)}(k_B)\big|_{\rm sB} = -P_{l,\eta=2,3}^{(z)}(-k_B)\big|_{\rm sB}. \tag{118}$$

We reach the conclusion that the polarizations for a strong magnetic field are conserved. This is an interesting result because it gives another way to talk about such symmetries, which is different from what has been developed in [15]. In fact, these polarizations are actually controlled by the external parameter B and can be adjusted by switching on B to different values.

9. Critical point analysis

As we have noted before in diagonalizing the Hamiltonian of the present system there is a critical point, i.e. $\lambda_{k,\sigma} = \pm\sqrt{\frac{1}{2}}$, with $\sigma = \pm 1$. It is a good task to study the effect of this point on the obtained results and underline when it will be new. To do this, we return to the Hamiltonian describing a system like harmonic oscillator where $\lambda_{k,\sigma}$ is included and use the corresponding spectrum to see what differences there are with respect to the standard case.

Taking into account such critical point, we can establish a relation between the Bloch wave vector and the Rashba one. More precisely, we have

$$k_{so}|_{\sigma=1} = \mp\sqrt{\frac{m\omega_c}{2\hbar}} - k_B, \qquad k_{so}|_{\sigma=-1} = \pm\sqrt{\frac{m\omega_c}{2\hbar}} + k_B. \tag{119}$$

It is convenient to write these relations as functions of the magnetic length l_B, such as

$$k_{so}|_{\sigma=1} = \mp\frac{1}{\sqrt{2}l_B} - k_B, \qquad k_{so}|_{\sigma=-1} = \pm\frac{1}{\sqrt{2}l_B} + k_B. \tag{120}$$

These relations are interesting in the sense that one vector can be determined in terms of the other. Moreover, they are B-dependent and clearly for a strong B one of them can be ignored whereas the remaining vector will be controlled by B. In this limit for instance the area of the Hall droplet can be written as

$$S = 2\pi l_B^2 \equiv \frac{\pi}{(k_{so}|_{\sigma=1})^2}. \tag{121}$$

Now let us return to write the corresponding Hamiltonian, which is

$$H_z^{(1)}\big|_{\pm\sqrt{\frac{1}{2}}} = \hbar\omega_c a^\dagger a. \tag{122}$$

This leads to the spectrum

$$\varepsilon^{(1)}_{l,n,\sigma}(k_B)\big|_{z,\pm\sqrt{\frac{1}{2}}} = \hbar\omega_c n, \qquad |n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle, \qquad n = 0, 1, 2 \ldots. \tag{123}$$

The first remark one can underline is that actually the Hilbert space of such a spectrum has an isolated point that corresponds to a zero mode energy. This is not a surprising result because there are some systems that behave like this. In fact, electrons in graphene, which behave like massless Dirac fermions (Majorana fermions), in the presence of a magnetic field have such kind of zero mode energy [19]. More precisely, one single particle in graphene can be described by the Dirac operator

$$H_{\mathsf{D}} = \frac{\mathrm{i}}{\sqrt{2}} v_{\mathsf{F}} \begin{pmatrix} 0 & d^\dagger \\ -d & 0 \end{pmatrix} \tag{124}$$

where $v_{\mathsf{F}} \approx \frac{c}{100}$ is the Fermi velocity, which will be set to one, and the many-body effects are neglected. The annihilation and creation operators, in the complex notation $z = x + \mathrm{i}y$, are given by

$$d = 2\frac{\partial}{\partial z} + \frac{B}{2}\bar{z}, \qquad d^\dagger = -2\frac{\partial}{\partial \bar{z}} + \frac{B}{2}z. \tag{125}$$

They verify the commutation relation

$$[d, d^\dagger] = 2B. \tag{126}$$

Now let us look at its square, which is

$$H_{\mathsf{D}}^2 = \frac{1}{2}\begin{pmatrix} d^\dagger d & 0 \\ 0 & dd^\dagger \end{pmatrix}. \tag{127}$$

Clearly, the first matrix element is similar to that of (122). The corresponding wavefunctions Ψ should be written in an appropriate form. This is

$$\Psi_{m,n} = \begin{pmatrix} \psi_{m,n} \\ \psi_{m-1,n} \end{pmatrix} \tag{128}$$

where the eigenfunctions $\psi_{m,n}$ are given by

$$\psi_{m,n}(z,\bar{z}) = \frac{(-1)^m\sqrt{B^m m!}}{\sqrt{2^{n+1}\pi(m+n)!}} z^n L_m^n\left(\frac{z\bar{z}}{2}\right) \mathrm{e}^{-\frac{B}{4}z\bar{z}}, \qquad m, n = 0, 1, 2 \ldots. \tag{129}$$

Their Landau levels take the form

$$E_{\mathrm{D}}^{2}(m) = Bm. \tag{130}$$

Finally, we end up with the zero mode energy for such systems, namely

$$\Psi^{(0,n)} = \begin{pmatrix} \psi_{0,n} \\ 0 \end{pmatrix}. \tag{131}$$

This is showing the similarity with the previous case that corresponds to the Hamiltonian $H_z^{(1)}$ at the critical point.

Now let us return to derive the whole spectrum in the present case. Indeed, as an immediate consequence the eigenvalues of $H^{(1)}$ becomes

$$\varepsilon_{l,n,\sigma}^{(1)}(k_B)\big|_{\pm\sqrt{\frac{1}{2}}} = \varepsilon_l^{(0)}(k_B + \sigma k_{so}) - \frac{\hbar^2 k_{so}^2}{2m} + \hbar\omega_c n. \tag{132}$$

The matrix element (56) now reads as

$$\langle n|k_z|n'\rangle = \pm\mathrm{i}\delta_{n,n'\pm 1}\sqrt{\frac{m\omega_c}{2\hbar}\left(n + \frac{1}{2} \mp \frac{1}{2}\right)}. \tag{133}$$

Focusing on the two first transverse modes, we show that different quantities forming the total spectrum take other forms. Indeed, we find

$$\varepsilon_l^{+}(k_B)\big|_{\pm\sqrt{\frac{1}{2}}} = \frac{1}{2}\left\{\varepsilon_l^{(0)}(k_B + k_{so}) + \varepsilon_l^{(0)}(k_B - k_{so})\right\} + \frac{\hbar\omega_c}{2} - \frac{\hbar^2 k_{so}^2}{2m} \tag{134}$$

as well as

$$\xi_l(k_B)|_{\pm\sqrt{\frac{1}{2}}} = \sqrt{\xi^2|_{\pm\sqrt{\frac{1}{2}}} + \left(\varepsilon_l^{-}(k_B) + \frac{\hbar\omega_c}{2}\right)^2} \tag{135}$$

where ξ at the critical point is given by

$$\xi|_{\pm\sqrt{\frac{1}{2}}} = \frac{\hbar^2 k_{so}}{m}\sqrt{\frac{m\omega_c}{2\hbar}}. \tag{136}$$

These leading to end up with the energies for different sub-bands, namely

$$\varepsilon_{l,\eta=1,2}(k_B) = \varepsilon_l^{+}(k_B) - \xi_l(k_B)|_{\pm\sqrt{\frac{1}{2}}}, \qquad \varepsilon_{l,\eta=3,4}(k_B) = \varepsilon_l^{+}(k_B) + \xi_l(k_B)|_{\pm\sqrt{\frac{1}{2}}}. \tag{137}$$

The corresponding Bloch spinors read as

$$\tilde{\theta}_{l,k_B,\eta=1,4}|_{\pm\sqrt{\frac{1}{2}}} = \begin{pmatrix} \frac{\mathrm{i}}{\xi|_{\pm\sqrt{\frac{1}{2}}}}\left[\varepsilon_l^{-}(k_B) + \frac{\hbar\omega_c}{2} + \xi_l(k_B)\big|_{\pm\sqrt{\frac{1}{2}}}\right] \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{138}$$

as well as for other modes

$$\tilde{\theta}_{l,k_B,\eta=2,3}|_{\pm\sqrt{\frac{1}{2}}} = \begin{pmatrix} 0 \\ -\frac{\mathrm{i}}{\xi|_{\pm\sqrt{\frac{1}{2}}}}\left[\varepsilon_l^{-}(k_B) + \frac{\hbar\omega_c}{2} - \xi_l(k_B)\big|_{\pm\sqrt{\frac{1}{2}}}\right] \\ 1 \\ 0 \end{pmatrix}. \tag{139}$$

The above materials can be used to analyze the polarization symmetries at the critical point. In fact, we show that for longitudinal and perpendicular cases the polarizations are zero. However it is finite along the z-direction and this can be traduced by the obtained results. Indeed, for $\eta = 1, \ldots, 4$ we obtain

$$P^{(z)}_{l,\eta}\big|_{\pm\sqrt{\frac{1}{2}}} = N^{-1}_{l,k_B,\eta=1,4}\big|_{\pm\sqrt{\frac{1}{2}}} \times \left\{ \frac{1}{\xi^2|_{\pm\sqrt{\frac{1}{2}}}} \left[\varepsilon_l^-(k_B) + \frac{\hbar\omega_c}{2} + \xi_l(k_B)|_{\pm\sqrt{\frac{1}{2}}} \right]^2 - 1 \right\} \tag{140}$$

and for $\eta = 2, 3$ we have

$$P^{(z)}_{l,\eta}\big|_{\pm\sqrt{\frac{1}{2}}} = N^{-1}_{l,k_B,\eta=2,3}\big|_{\pm\sqrt{\frac{1}{2}}} \times \left\{ 1 - \frac{1}{\xi^2|_{\pm\sqrt{\frac{1}{2}}}} \left[\varepsilon_l^-(k_B) + \frac{\hbar\omega_c}{2} - \xi_l(k_B)|_{\pm\sqrt{\frac{1}{2}}} \right]^2 \right\}. \tag{141}$$

These allow us to end up with the relation

$$P^{(z)}_{l,\eta=1,4}(k_B)|_{\pm\sqrt{\frac{1}{2}}} \neq -P^{(z)}_{l,\eta=2,3}(-k_B)|_{\pm\sqrt{\frac{1}{2}}} \tag{142}$$

which means that there is no preservation of such symmetries as we have seen in the standard case. With this we end the analysis of the four cases and we will see the last one.

10. Periodic structure without the magnetic field

To complete the present analysis it is important to treat the case of the periodic structure without the magnetic field. This will allow us to get new results and make contact with other proposals. In doing so, we simply return to the total Hamiltonian and derive the corresponding spectrum when B is zero.

To start let us consider the Hamiltonian describing the system in the absence of B. This can be obtained from (11) to end up with

$$H|_{B=0} = \frac{\hbar^2\vec{k}^2}{2m} - \frac{\hbar^2 k_{so}}{m}(\sigma_x k_z - \sigma_z k_x) + U(x). \tag{143}$$

Clearly for arbitrary periodic potential $U(x)$, the eigenvalues (35) and eigenstates (36) reduce to those of wave planes. Indeed, it is easy to note

$$|n\rangle \longrightarrow |k_z\rangle, \qquad \varepsilon^{(1)}_{l,n,\sigma}(k_B)|_z \longrightarrow \frac{\hbar^2 k_z^2}{2m}. \tag{144}$$

These can be used to show that the matrix element takes the form

$$\langle k_z|k_z|k_z'\rangle = \mathrm{i}\delta_{k_z,k_z'}k_z. \tag{145}$$

Combining all together to get the energies corresponding to different sub-bands. Note that we have now only two possible values for η, i.e. (1,2). Thus, we have

$$\varepsilon_{l,\eta=1,2}(k_B,k_z) = \frac{1}{2}\left[\varepsilon_l^{(0)}(k_B+k_{so}) + \varepsilon_l^{(0)}(k_B-k_{so})\right] + \frac{\hbar^2k_z^2}{2m} - \frac{\hbar^2k_{so}^2}{2m}$$
$$\pm\sqrt{\left(\varepsilon_l^-(k_B)\right)^2 + \left(\frac{\hbar^2 k_{so}k_z}{m}\right)^2}. \tag{146}$$

At this point, there are two limits and these should be discussed to characterize the present situation. The first one is concerning the case where we assume that $k_B = 0$ and $k_z > 0$. Returning to the spectrum to obtain

$$\varepsilon^{2D}_{l,\eta=1,2}(k_B=0,k_z) = \varepsilon_l^{(0)}(k_{so}) + \frac{\hbar^2k_z^2}{2m} - \frac{\hbar^2k_{so}^2}{2m} \pm \frac{\hbar^2k_{so}k_z}{m}. \tag{147}$$

This tell us that the energy branch with $\eta = 2$ has a minimum at $k_z = k_{so}$ for all band l. These analytical results are in contrast to what was numerically predicted in [20].

The second limit can be fixed by requiring that $U(x) = 0$. In this case, one can show that the corresponding energies are

$$\varepsilon_{\sigma,\eta=1,2}(k_x, k_z) = \frac{\hbar^2}{2m}\left(k_x^2 + k_z^2 + 2\sigma k_{so}\sqrt{k_x^2 + k_z^2}\right). \tag{148}$$

In conclusion (148) is in agreement with the Rashba energy for the present system without periodic potential and magnetic field.

11. Conclusion

We have investigated the basic features of two-dimensional periodic structures with Rashba spin–orbit interaction in the presence of an external magnetic field B. This latter feature allowed us to end up with a confining potential along the z-direction that has been used to deal with different issues. In particular, it has been served to split the corresponding Hamiltonian into two parts and one of them has resulted in behaving like a one-dimensional harmonic oscillator. This decomposition was useful in the sense that different spectrums are obtained leading to the total one. To make comparisons with other proposals, we have restricted ourselves to the two first transverse modes. These are used to derive the eigenvalues and the Bloch spinors for four sub-bands $\eta = 1, \ldots, 4$. Looking at the obtained spectrum we have shown that there is a hidden symmetry, which has been used to talk about the polarizations.

After getting the eigenvalues and the corresponding eigenspinors, we have analyzed the polarization symmetries by evaluating their components in different directions. As a consequence, the longitudinal and perpendicular cases are obtained to be nulls except along the z-direction where a finite value is found. Looking at the symmetry with respect to the Bloch vector, we have shown that the presence of the magnetic field breaks such symmetries in disagreement with the results obtained in [15]. As another task, we have investigated the dynamical spin by determining some relevant quantities used to write down the appropriate spin Hall conductivity.

Subsequently, different limits have been studied. Indeed, we have analyzed the case when the Rashba coupling is null and this allowed us to end up with a Landau problem submitted to the periodicity constraint. Therefore, the spectrum is easily obtained by noting that the corresponding energies are degenerates and their degree of degeneracy can be determined in terms of the Bloch vector. Moreover, we have considered another case that is $U(x) = 0$, which leads to the Landau problem being amended with the Rashba coupling in the presence of the magnetic field. Solving the eigenvalue equation we have derived the spectrum that has been used to conclude that the polarizations are not preserved.

Inspecting other limits, the strong magnetic field case has been considered. This allowed us to get different conclusions and derive interesting results. Indeed, after getting the corresponding spectrum, we have tackled the symmetries problem. More precisely, we have shown that in the present situation the polarizations are conserved and therefore offered another way to talk about such symmetries. More importantly, such analysis allowed us to make contact with the proposal developed in [15] and in particular to show how one can recover the symmetry preservations from the present work. Moreover, the obtained polarizations are magnetic field dependent and therefore it can be adjusted by switching the field to any value.

On the other hand, we have studied in detail one important issue that allowed us to make contact with other systems. In fact, the critical point $\lambda_{k,\sigma} = \pm\sqrt{\frac{1}{2}}$ that appeared as a singularity in diagonalizing the Hamiltonian $H_z^{(1)}$ has been analyzed. Using the corresponding spectrum we have ended up with a zero mode energy in the Hilbert space of $H_z^{(1)}$. In fact, a comparison to the Dirac fermions in the presence of the magnetic field is established.

Finally, to get more information about the present system we have investigated the case where B is switched off. This has been done by deriving the spectrum for $\eta = 1, 2$, which are used to study two limits. Indeed, assuming that $k_B = 0$ where $k_z > 0$, we have shown that the energy for $\eta = 2$ has a minimum at $k_z = k_{so}$ for all band l, which is in agreement with what has been numerically predicted in [20]. Moreover, requiring that $U(x) = 0$, we have recovered the Rashba energy.

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